2.

$$\left.\begin{array}{l} x + y + z + t = 1 \\ -ax + y + z + t = 2 \\ x - ay + z + t = 3 \\ x + y - az + t = 4 \\ x + y + z - at = 5 \end{array}\right\}$$

$$\begin{array}{c} A \qquad\qquad A^* \\ \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -a & 1 & 1 & 1 & 2 \\ 1 & -a & 1 & 1 & 3 \\ 1 & 1 & -a & 1 & 4 \\ 1 & 1 & 1 & -a & 5 \end{array}\right) \end{array} \xrightarrow[\substack{f_5 - f_1 \\ f_4 - f_1 \\ f_3 - f_1 \\ f_2 - f_1}]{} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -a-1 & 0 & 0 & 0 & 1 \\ 0 & -a-1 & 0 & 0 & 2 \\ 0 & 0 & -a-1 & 0 & 3 \\ 0 & 0 & 0 & -a-1 & 4 \end{array}\right) \quad -a-1=0 \rightarrow a=-1$$

$a = -1 \rightarrow Rg(A) = 1$, $Rg(A^*) = 2 \rightarrow$ S. I.

$a \neq -1 \rightarrow Rg(A) = 4$, $Rg(A^*) = 5 \rightarrow$ S. I.

Es un S.I. para todo valor de a.

3. a) $L_1 = [(a, 1, -1, 2), (1, b, 0, 3)]$ $\quad L_2 = [(1, -1, 1, -2), (-2, 0, 0, -6)]$

$$\begin{pmatrix} 1 & -1 & 1 & -2 \\ -2 & 0 & 0 & -6 \\ a & 1 & -1 & 2 \end{pmatrix} \xrightarrow[f_3 + f_1]{} \begin{pmatrix} 1 & -1 & 1 & -2 \\ -2 & 0 & 0 & -6 \\ a+1 & 0 & 0 & 0 \end{pmatrix} \rightarrow a = -1 //$$

$$\begin{pmatrix} 1 & -1 & 1 & -2 \\ -2 & 0 & 0 & -6 \\ 1 & b & 0 & 3 \end{pmatrix} \xrightarrow[f_3 + \frac{1}{2} f_2]{} \begin{pmatrix} 1 & -1 & 1 & -2 \\ -2 & 0 & 0 & -6 \\ 0 & b & 0 & 0 \end{pmatrix} \rightarrow b = 0 //$$

b) $W_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \underbrace{M_{2\times 2}(\mathbb{R})}_{\dim 4} \,/\, \underbrace{a + b + c + d = 0}_{\text{1 ec li}} \right\}$ $\quad \dim W_1 = 4 - 1 = 3$

$a = \alpha \quad b = \beta \quad c = \lambda \quad d = -(\alpha + \beta + \lambda) \quad \alpha, \beta, \lambda \in \mathbb{R}$

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \lambda & -(\alpha+\beta+\lambda) \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) : \alpha, \beta, \lambda \in \mathbb{R} \right\} \rightarrow \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$L \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\} //$$

$$W_2 = L \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \rightarrow \alpha \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \alpha, \beta \in \mathbb{R}$$

$$\underbrace{a = \alpha \quad b = \alpha}_{a - b = 0} \;\; \underbrace{\quad c = \alpha}_{b - c = 0} \qquad d = \alpha + \beta$$

c) $W_1 = \{(x, y, z, t) \in \mathbb{R}^4 / x - y + 3z = 0,\ x + y - 2z + 2t = 0\}$

$x = \alpha$

$y = \beta$

$z = \frac{\beta - \alpha}{3}$

$t = \frac{\alpha + \beta - 2\left(\frac{\beta - \alpha}{3}\right)}{-2} = \frac{-\beta - 5\alpha}{6}$

$\left\{\left(\alpha, \beta, \frac{\beta - \alpha}{3}, \frac{-\beta - 5\alpha}{6}\right) \in \mathbb{R}^4 / \alpha, \beta \in \mathbb{R}\right\}$

$L = \left\{\left(1, 0, -\frac{1}{3}, -\frac{5}{6}\right), \left(0, 1, \frac{1}{3}, -\frac{1}{6}\right)\right\}$ //

$\forall \vec{u}, \vec{v} \in W_1 \to \vec{u} + \vec{v} \in W_1$

$\forall \vec{u} \in W_1, \forall \alpha \in \mathbb{R} \to \alpha \vec{u} \in W_1$

$\vec{u} = (u_1, u_2, u_3, u_4) \to u_1 - u_2 + 3u_3 = 0,\ u_1 + u_2 - 2u_3 + 2$

$\vec{v} = (v_1, v_2, v_3, v_4) \to v_1 - v_2 + 3v_3 = 0,\ v_1 + v_2 - 2v_3 + 2$

$(u_1 + v_1) - (u_2 + v_2) + 3(u_3 + v_3) = 0 \to \underbrace{u_1 - u_2 + 3u_3}_{0} + \underbrace{v_1 - v_2 + 3v_3}_{0} = 0$ ✓

$\alpha u_1 - \alpha u_2 + 3\alpha u_3 = 0 \to \alpha(u_1 - u_2 + 3u_3) = \alpha \cdot 0 = 0$ ✓

4. a) $f(-1, 1, 3) = (6, -4, 16)$
$f(-2, 1, 1) = (-2, -5, 1)$
$f(3, 2, -1) = (1, 14, -12)$

$f(1,0,0) = a(-1,1,3) + b(-2,1,1) + c(3,2,-1) \to$

$\left.\begin{array}{l} -a - 2b + 3c = 1 \\ a + b + 2c = 0 \\ 3a + b - c = 0 \end{array}\right\} \begin{array}{l} f_2 + f_1 \\ f_3 + 3f_1 \end{array} \left\{\begin{array}{l} -a - 2b + 3c = 1 \\ -b + 5c = 1 \\ -5b + 8c = 3 \end{array}\right\} 5f_2 - f_3 \to c = \frac{2}{17}$

$\to b = \frac{-7}{17}$

$\to a = \frac{3}{17}$

$\frac{3}{17}(6, -4, 16) - \frac{7}{17}(-2, -5, 1) + \frac{2}{17}(1, 14, -12) = (2, 3, 1)\ *_1$

$f(0,1,0) = a(-1,1,3) + b(-2,1,1) + c(3,2,-1) \to a = \frac{-1}{17}\ b = \frac{8}{17}\ c = \frac{5}{17}$

$\frac{-1}{17}(6, -4, 16) + \frac{8}{17}(-2, -5, 1) + \frac{5}{17}(1, 14, -12) = (-1, 2, -5)\ *_2$

$f(0,0,1) = a(-1,1,3) + b(-2,1,1) + c(3,2,-1) \to a = \frac{7}{17}\quad b = \frac{-5}{17}\quad c = \frac{-1}{17}$

$\frac{7}{17}(6, -4, 16) - \frac{5}{17}(-2, -5, 1) - \frac{1}{17}(1, 14, -12) = (3, -1, 7)\ *_3$

$M_{B_c \to B_c}(f) = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & -5 & 7 \end{pmatrix}$

b) $\left.\begin{array}{l} 2x + 3y + z = 0 \\ -x + 2y - 5z = 0 \\ 3x - y + 7z = 0 \end{array}\right\} \left\{\begin{array}{l} x = 0 \\ y = 0 \\ z = 0 \end{array}\right.$ SCD $(Rg(A) = Rg(A^*) = n°\ \text{incognitas}) \to B\ Ker f = (0,0,0) = \emptyset$

$B\ Im f = \{(2, -1, 3), (3, 2, -1), (1, -5, 7)\} = \mathbb{R}^3(f)$.

c) $Ker f = 0 \to$ Inyectiva ✓
$Im f = \mathbb{R}^3 \to$ Sobreyectiva ✓ } Biyectiva ✓

5 a) $B = \{u_1, u_2, u_3, u_4\} = \{(2,1,0,2), (2,-2,1,0), (1,1,1,1), (5,0,2,3)\}$

$V_1 = (2,1,0,2)$

$V_2 = u_2 - \frac{\langle u_2 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 = (2,-2,1,0) - \frac{(2,-2,1,0) \times (2,1,0,2)}{(2,1,0,2) \times (2,1,0,2)} \cdot (2,1,0,2) =$

$(2,-2,1,0) - \frac{2}{9}(2,1,0,2) = \left(\frac{14}{9}, -\frac{20}{9}, 1, -\frac{4}{9}\right)$

$V_3 = u_3 - \frac{\langle u_3 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 - \frac{\langle u_3 \cdot V_2 \rangle}{\langle V_2 \cdot V_2 \rangle} \cdot V_2 = (1\,1\,1\,1) - \frac{5}{9}(2\,1\,0\,2) + \frac{1}{77}\left(\frac{14}{9}, -\frac{20}{9}, 1, \frac{-4}{9}\right) =$

$V_3 = \left(-\frac{1}{11}, \frac{32}{77}, \frac{78}{77}, -\frac{9}{77}\right)$

$V_4 = (0,0,0,0)$ No se usa

$V_1 = (2,1,0,2) \to V'_1 = \left(\frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}\right)$

$V_2 = \left(\frac{14}{9}, -\frac{20}{9}, 1, \frac{-4}{9}\right) \to V'_2 = \left(\frac{2}{33}\sqrt{77}, -\frac{20}{231}\sqrt{77}, \frac{3}{77}\sqrt{77}, -\frac{4}{231}\sqrt{77}\right)$

$V_3 = \left(-\frac{1}{11}, \frac{32}{77}, \frac{78}{77}, -\frac{9}{77}\right) \to V'_3 = \left(\frac{-1}{1034}\sqrt{7238}, \frac{16}{3619}\sqrt{7238}, \frac{39}{3619}\sqrt{7238}, \frac{-9}{7238}\sqrt{7238}\right)$

b) $u_1 = (2,1,0\,2) \to u'_1 = \left(\frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}\right)$

$u_2 = (2,-2,1,0) \to u'_2 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, 0\right)$

$u_3 = (1,1,1,1) \to u'_3 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$W = (5,0,2,3)$

$W' = \langle W \cdot u'_1 \rangle u'_1 + \langle W \cdot u'_2 \rangle u'_2 + \langle W \cdot u'_3 \rangle u'_3 = \left(\frac{20}{9}, 0, 0, \frac{20}{9}\right) + \left(\frac{20}{9}, 0, \frac{2}{9}, 0\right) + \left(\frac{5}{4}, 0, \frac{1}{2}, \frac{3}{4}\right)$

$W' = \left(\frac{205}{36}, 0, \frac{13}{18}, \frac{107}{36}\right)$

6 $A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & m & 8 \\ 2 & 8 & m \end{pmatrix}$ $|A - \lambda I| \to (1-\lambda)(m-\lambda)^2 - 64(1-\lambda) = (m^2 + \lambda^2 - 2m\lambda)(1-\lambda) - 64(1-\lambda) =$

$m^2 + \lambda^2 - 2m\lambda - \lambda m^2 - \lambda^3 + 2m\lambda^2 - 64 + 64\lambda =$

$-\lambda^3 + (2m+1)\lambda^2 + (-2m - m^2 + 64)\lambda + (m^2 - 64) = 0$

$$A' = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 8 \\ 2 & 8 & 0 \end{pmatrix} \quad |A' - \lambda I| = \lambda^2(1-\lambda) - 64(1-\lambda) = -\lambda^3 + \lambda^2 + 64\lambda - 64 = 0$$

$$(\lambda = 1 \quad \lambda = 8 \quad \lambda = -8) \rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 8 \\ 0 & 0 \end{pmatrix}$$

$$\lambda = 1: \quad A' - \lambda I = \begin{pmatrix} 0 & -2 & -2 \\ -2 & -1 & 8 \\ 2 & 8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow x = -\frac{9}{2}\alpha \quad y = \alpha \quad z = -\alpha \qquad \alpha = 2$$

$$\lambda = 8: \quad A' - \lambda I = \begin{pmatrix} -7 & -2 & -2 \\ -2 & -8 & 8 \\ 2 & 8 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow x = \frac{-8}{13}\alpha \quad y = \frac{15}{13}\alpha \quad z = \alpha \qquad \alpha = 13$$

$$\lambda = -8 \quad A' - \lambda I = \begin{pmatrix} 9 & -2 & -2 \\ -2 & 8 & 8 \\ 2 & 8 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow x = 0 \quad y = -\alpha \quad z = \alpha \qquad \alpha = 1$$

$$Q = \begin{pmatrix} -9 & -8 & 0 \\ 2 & 15 & -1 \\ -2 & 13 & 1 \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} -\frac{1}{9} & -\frac{2}{63} & -\frac{2}{63} \\ 0 & \frac{1}{28} & \frac{1}{28} \\ -\frac{2}{9} & -\frac{19}{36} & \frac{17}{36} \end{pmatrix} \rightarrow A' = QDQ^{-1} \checkmark$$

1. a) Verdadero, porque el rango de A y A^* serán iguales entre sí y distinto del nº de incognitas

b) Verdadero, porque el sistema generador debe reunir las características del subespacio que genera.

c) Falso, porque si fuese lineal se debe cumplir esta situación:

$$f(A) \rightarrow A^{-1} \qquad f(A+B) = f(A) + f(B)$$

$$f(B) \rightarrow B^{-1} \qquad (A+B)^{-1} \neq A^{-1} + B^{-1}$$

d) Falso. Basta con que sea ortogonal.

e) Verdadero, ya que tienen el mismo polinomio característico y los mismos autovalores.